Orthogonal Projection onto a Subspace Spanned by a Nonorthogonal Basis

Let $n_1, \dots, n_m \in \mathbb{R}^d$ be a basis of dim $m \leq d$.

Problem 1.

Given $v \in \mathbb{R}^d$, find $v' \in \mathbb{R}^d$ such that

$$v' = \sum_i \lambda_i n_i \qquad (1a)$$

and

$$(v' - v) \cdot n_i = 0 \qquad (1b)$$

for all $i = 1, \dots, m$.

With $N = (n_1, \dots, n_m) \in \mathbb{R}^{d \times m}$ we reformulate, find $\lambda \in \mathbb{R}^m$ such that

$$v' = N\lambda \quad \Rightarrow \quad N^T (v' - v) = 0. \qquad (2)$$

Combining the equation yields

$$N^T (N\lambda - v) = 0 \quad \Rightarrow$$

$$\lambda = (N^T N)^{-1} N^T v. \qquad (3)$$

With the Gramian $G = N^T N \in \mathbb{R}^{m \times m}$ we write the solution (and also verify uniqueness of λ) as

$$v' = P_N v, \qquad P_N = N G^{-1} N^T. \qquad (4)$$

P_N is an orthogonal projector ($P_N^2 = P_N$, $P_N^T = P_N$).

In the special case of orthonormal n_i we obtain the simplification $P_N = NN^T$.

The complement operation can be written as

Problem 2.

Find the vector v'' with $v'' \cdot n_i = 0$ that minimizes $\|v'' - v\|_2$.

The solution is

$$v'' = P_C v, \qquad P_C = 1 - P_N = 1 - N G^{-1} N^T \qquad (5)$$

This result also follows by solving the constrained optimization problem through the Lagrangian functional

$$\mathcal{L} = \frac{1}{2} \|v'' - v\|_2^2 - \sum_i \lambda_i n_i \cdot v'' . \qquad (6)$$

In words, we compute the vector closest to v that is orthogonal to the given normal vectors n_i, even if they are not orthogonal.