

ON THE NUMERICAL EVALUATION OF FRACTIONAL SOBOLEV NORMS

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ABSTRACT. In several important and active fields of modern applied mathematics, such as the numerical solution of PDE-constrained control problems or various applications in image processing and data fitting, the evaluation of (integer and real) Sobolev norms constitutes a crucial ingredient. Different approaches exist for varying ranges of smoothness indices and with varying properties concerning exactness, equivalence and the computing time for the numerical evaluation. These can usually be expressed in terms of discrete Riesz operators.

We propose a collection of criteria which allow to compare different constructions. Then we develop a unified approach which is valid for non-negative real smoothness indices for standard finite elements, and for positive and negative real smoothness for biorthogonal wavelet bases. This construction delivers a wider range of exactness than the currently known constructions and is computable in linear time.

1. Introduction. Sobolev norms are a fundamental mathematical concept which is widely used for modern research in the areas – among many others – of applied mathematics and numerical simulation. They arise in a natural way in the weak formulation of partial differential and integral equations, as regularization terms in image processing and data fitting, and in the objective functional of PDE-constrained optimal control problems. There exists an abundant collection of literature on these subjects, see for example [2, 7, 13, 30, 32].

However, the role of equivalences between Sobolev norms in the functional analytic setting and discrete norms on coefficient spaces as a central building block for the theory and practice of fast solvers has matured only during the last 10 to 15 years. Such norm equivalences relate to the approximate evaluation of norms and the development of optimal preconditioners for partial differential equations, such as multigrid [8, 29, 39], BPX-type [9, 41, 42], and wavelet schemes [14, 18, 24, 25], see also [26, 27, 35].

For these applications, mostly the order of smoothness is relevant, and not the actual value of a norm. Estimations of equivalence up to uniformly bounded constant factors are generally sufficient to establish theoretical results. Therefore, computations up to spectral equivalence are also common in practice. These can be

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performed via the application of Riesz matrices and multilevel scaling operations, which are approximate discrete representations of operators in function space. —

Recently, the focus of interest is somewhat shifting from the solution of a single partial differential equation towards more complex problem formulations. Motivated by industrial applications, the subject of optimization subject to PDE constraints is quickly gaining importance, see e.g. [4, 5, 28, 38]. For such control problems, a penalty functional which usually contains weighted sums of Sobolev norms has to be minimized, where the physical model represented by a set of PDEs must be respected as a constraint. To compare different models and simulations, and to satisfy the objectives often given by applications in engineering, it is important that the numerical modeling of the norms under consideration obeys a minimal set of criteria.

These applications require in the first place that norms of integral smoothness, i.e., on the spaces L_2 or H^1 , are computed exactly up to discretization error. To preserve the optimal runtime complexity of state-of-the-art numerical methods, also the evaluation of norms must be performed in a time which is proportional to the number of unknowns. Thus, the fast computation is a crucial requirement from the practical point of view.

There are also applications which motivate to consider negative or positive real norms. This is also interesting from a purely mathematical point of view. For example, control problems with boundary control naturally lead to norms on $H^{\frac{1}{2}}$ or its dual, and the weak formulation of elliptic PDEs and integral equations motivates to consider functions in $(H^1)'$. As there exist different definitions of non-integer Sobolev norms, it is only reasonable to demand evaluation up to equivalence here.

Thus, a consistent modeling implies that equivalence for all occurring smoothness indices is a necessity. Moreover, we propose as a further criterion that the norm of constant functions – which reduces to the L_2 norm – shall be evaluated exactly. We have included this seemingly obvious criterion since it is by no means respected by all existing approaches.

Having motivated this set of criteria, it remains to state that there are no constructions with finite elements or wavelets we know of which satisfy all of them (this statement will become more clear later in this document). The numerical evaluation of Sobolev norms in this more general context has to our knowledge not yet been systematically studied. (We exclude spectral elements [6] here, as the compact support of all basis functions and non-periodic boundary conditions are important for the applications we have in mind.)

In this paper, we present a framework which realizes the complete set of criteria for finite elements and non-negative real order of smoothness, and for biorthogonal wavelets and arbitrary smoothness. It is derived from the concept of additive multilevel preconditioners in the spirit of BPX-type and wavelet methods. A sketch of the general idea restricted to the wavelet context was given in [10]. We realize here a suitable construction for finite elements, and improve the original wavelet-specific construction. This new approach is largely similar in nature for the realms of finite elements and wavelets, which justifies in our opinion to call it a unified scheme for the numerical evaluation of general, that is, integer and real Sobolev norms. —

This paper is organized as follows. In Section 2, we cover the theoretical basis for the subjects of multiresolution analysis and biorthogonality, which are essential for the development of our construction of Riesz operators. Section 3 is dedicated to the development of our scheme in the finite element context. The general construction

for primal and dual norms is realized in the wavelet setting in Section 4, using the full theory of biorthogonal space decompositions.

Let us now comment on some notational conventions. To emphasize the role of duality, we will write $\langle v, \tilde{v} \rangle$ for the dual pairing of two functions $v \in \mathcal{H}$, $\tilde{v} \in \mathcal{H}'$, where \mathcal{H} denotes a general Hilbert space with dual \mathcal{H}' . For the special case $\mathcal{H} = \mathcal{H}' = L_2$, this expression reduces to the standard inner product $(v, \tilde{v})_{L_2}$. For collections of functions $\Phi = \{\phi\} \subset \mathcal{H}$, which we consistently interpret as (possibly infinite) column vectors, and some single function $f \in \mathcal{H}'$, the term $\langle \Phi, f \rangle$ is to be understood as a column vector, and $\langle f, \Phi \rangle$ as a row vector, according to

$$\langle \Phi, f \rangle := (\langle \phi, f \rangle)_{\phi \in \Phi}, \quad \langle f, \Phi \rangle := \langle \Phi, f \rangle^T.$$

The relations for the dual functions are defined analogously. Consequently, by-element dual pairings of two function sequences $\Phi \subset \mathcal{H}$ and $\tilde{\Phi} = \{\tilde{\phi}\} \subset \mathcal{H}'$ are written as (not necessarily square and possibly infinite) matrices $\langle \Phi, \tilde{\Phi} \rangle$ with entries

$$\langle \Phi, \tilde{\Phi} \rangle := (\langle \phi, \tilde{\phi} \rangle)_{\phi \in \Phi, \tilde{\phi} \in \tilde{\Phi}}.$$

It follows that linear transformations of two function sequences $\Phi, \tilde{\Phi}$ by (not necessarily square) matrices \mathbf{A} and \mathbf{B} satisfy

$$\langle \mathbf{A}\Phi, \mathbf{B}\tilde{\Phi} \rangle = \mathbf{A}\langle \Phi, \tilde{\Phi} \rangle \mathbf{B}^T.$$

We specify by H^s the Sobolev space of (possibly fractional) index of smoothness $s \in \mathbb{R}$ over a domain $\Omega \subset \mathbb{R}^n$, and abbreviate its dual by $H^{-s} := (H^s)'$. (It would be equally valid to consider the space H_0^s , since the following derivations are general with respect to boundary conditions).

The relation $a \sim b$ means that $ca \leq b \leq Ca$ with constants c, C which are independent of any parameters on which a or b may depend.

2. General considerations. This section covers the essential theory. Most of it can be found in literature on wavelet methods for PDEs, such as [18].

2.1. Multiresolution. The concept of *multiresolution analysis* is central for (multilevel) finite elements and (biorthogonal B-spline) wavelet bases alike, see e.g. [23, 33, 34]. We collect some basic facts here and introduce our notation and conventions.

Definition 2.1. A *multiresolution sequence* $\mathcal{S} = \{S_j\}_{j \geq j_0}$ over the Hilbert space $\mathcal{H} = L_2(\Omega)$, $\Omega \subset \mathbb{R}^n$, is a set of nested closed subspaces $S_j \subset \mathcal{H}$ with the following properties,

$$S_{j_0} \subset S_{j_0+1} \subset \dots \subset \mathcal{H}, \quad \text{clos}_{\mathcal{H}} \left(\bigcup_{j \geq j_0} S_j \right) = \mathcal{H}, \quad (1)$$

where $j_0 \in \mathbb{Z}$ denotes the coarsest level of resolution.

The subspaces S_j typically have the form

$$S_j = \text{clos}_{\mathcal{H}}(\text{span}(\Phi_j)) \quad \text{with} \quad \Phi_j := \{\phi_{j,k} : k \in \Delta_j\}, \quad (2)$$

where $\phi_{j,k}$ are suitable basis functions for S_j over corresponding (possibly infinite) index sets Δ_j with cardinality

$$N_j := \#\Delta_j, \quad \bar{N}_J := \sum_{j=j_0}^J \#N_j, \quad N_j, \bar{N}_J \sim \rho^{nj},$$

and the practically most relevant case being $\rho = 2$.

The *generator bases* or *single-scale bases* $\{\Phi_j\}_{j \geq j_0}$ defined in (2) are chosen to be *uniformly stable* in the sense that

$$\|\mathbf{c}\|_{\ell_2(\Delta_j)} \sim \|\mathbf{c}^T \Phi_j\|_{\mathcal{H}} \quad (3)$$

for any coefficient vector $\mathbf{c} \in \ell_2(\Delta_j)$. As a special case it follows that the basis functions are normed in \mathcal{H} ,

$$\|\phi_{j,k}\|_{\mathcal{H}} \sim 1.$$

In the following, we will drop the subscript on norms over the sequence spaces ℓ_2 , simply writing $\|\cdot\|$, unless we wish to emphasize the choice of the index set. For numerical purposes, it is important that the basis functions satisfy the *locality condition*

$$\text{diam supp}(\phi_{j,k}) \sim \rho^{-j}. \quad (4)$$

As the spaces S_j are nested, the generator functions of adjacent scales obey the *two-scale relations*

$$\Phi_j^T = \Phi_{j+1}^T \mathbf{G}_{j,0}, \quad (5)$$

which define certain $N_{j+1} \times N_j$ *refinement matrices* $\mathbf{G}_{j,0} : \ell_2(\Delta_j) \rightarrow \ell_2(\Delta_{j+1})$. As a consequence of locality (4), the matrices $\mathbf{G}_{j,0}$ are *uniformly sparse*, that is, the number of entries per row and column is uniformly bounded by a constant.

Generator functions on coarse levels j can be expressed by functions on any finer level J by the iterated use of (5),

$$\Phi_j^T = \Phi_J^T \mathbf{T}_{J,j} \quad \text{with} \quad \mathbf{T}_{J,j} := \mathbf{G}_{J-1,0} \mathbf{G}_{J-2,0} \cdots \mathbf{G}_{j,0}, \quad \mathbf{T}_{J,J} := \mathbf{I}. \quad (6)$$

Collecting the basis functions from all levels results in a redundant set or *frame* Θ_J ,

$$\Theta_J^T := (\Phi_{j_0}^T, \dots, \Phi_{J-1}^T, \Phi_J^T) = \Phi_J^T (\mathbf{T}_{J,j_0}, \dots, \mathbf{T}_{J,J-1}, \mathbf{T}_{J,J}) =: \Phi_J^T \mathbf{H}_J, \quad (7)$$

with the rectangular $N_J \times \bar{N}_J$ transformation matrix \mathbf{H}_J .

The notion of *complement spaces* plays an important role in the context of multilevel preconditioning and norm equivalences. It follows from (1) that there exist sequences $\Psi_j := \{\psi_{j,k} : k \in \nabla_j\}$ such that

$$S_{j+1} = S_j \oplus W_j \quad \text{with} \quad W_j := \text{clos}_{\mathcal{H}}(\text{span}(\Psi_j)). \quad (8)$$

When an explicit specification of the *complement basis* Ψ_j is needed, it is usually selected such that the collections $\Phi_j \cup \Psi_j$ are uniformly stable in the sense of (3) for all $j \geq j_0$. Then it follows further that there exist matrices $\mathbf{G}_{j,1} : \ell_2(\nabla_j) \rightarrow \ell_2(\Delta_{j+1})$ satisfying

$$\Psi_j^T = \Phi_{j+1}^T \mathbf{G}_{j,1}. \quad (9)$$

It is important for us that the basis Ψ_j also has compact support analogous to (4), which causes $\mathbf{G}_{j,1}$ to be uniformly sparse as well. The complete two-scale relations read

$$(\Phi_j^T, \Psi_j^T) = \Phi_{j+1}^T \mathbf{G}_j \quad \text{with} \quad \mathbf{G}_j := (\mathbf{G}_{j,0}, \mathbf{G}_{j,1}) : \ell_2(\Delta_{j+1}) \rightarrow \ell_2(\Delta_{j+1}). \quad (10)$$

From (8) it follows that \mathbf{G}_j is invertible. Moreover, $\Phi_j \cup \Psi_j$ is uniformly stable if and only if

$$\|\mathbf{G}_j\| \sim 1, \quad \|\mathbf{G}_j^{-1}\| \sim 1.$$

2.2. Mass, Stiffness and Riesz Matrices. The generator bases Φ_j for the Hilbert space $\mathcal{H} = L_2(\Omega)$ may for example be variants of the nodal basis, spline bases or constructions based on polynomials, or multivariate extensions thereof, provided that uniform stability holds, i.e., (3) is satisfied.

This section is concerned with the computation of inner products. Let $v_1, v_2 \in S_j \subset L_2$ be two functions with expansion coefficients $\mathbf{c}_1, \mathbf{c}_2 \in \ell_2$, that is, $v_i = \mathbf{c}_i^T \Phi_j$. Clearly, their L_2 inner product can be evaluated according to

$$(v_1, v_2)_{L_2} = (\mathbf{c}_1^T \Phi_j, \mathbf{c}_2^T \Phi_j)_{L_2} = \mathbf{c}_1^T (\Phi_j, \Phi_j)_{L_2} \mathbf{c}_2 =: \mathbf{c}_1^T \mathbf{M}_j \mathbf{c}_2 \quad (11)$$

with the *mass matrix* $\mathbf{M}_j = (\Phi_j, \Phi_j)_{L_2}$. It is symmetric and positive definite. By (3), its spectral condition number $\kappa(\mathbf{M}_j)$ is uniformly bounded,

$$\mathbf{c}^T \mathbf{M}_j \mathbf{c} = (\mathbf{c}^T \Phi_j, \mathbf{c}^T \Phi_j)_{L_2} = \|\mathbf{c}^T \Phi_j\|_{L_2}^2 \sim \|\mathbf{c}\|^2 = \mathbf{c}^T \mathbf{c} \implies \kappa(\mathbf{M}_j) \sim 1.$$

Thus, the L_2 norm of any function $v = \mathbf{c}^T \Phi_j \in S_j$ can be computed via the mass matrix,

$$\|v\|_{L_2}^2 = \mathbf{c}^T \mathbf{M}_j \mathbf{c}.$$

The computation of higher order integral Sobolev norms is analogous. Consider for example the H^1 seminorm,

$$|v|_{H^1}^2 = (\nabla v, \nabla v)_{L_2} = (\nabla \mathbf{c}^T \Phi_j, \nabla \mathbf{c}^T \Phi_j)_{L_2} = \mathbf{c}^T (\nabla \Phi_j, \nabla \Phi_j) \mathbf{c} =: \mathbf{c}^T \mathbf{A}_j \mathbf{c},$$

where $\mathbf{A}_j = (\nabla \Phi_j, \nabla \Phi_j)_{L_2}$ is the standard finite element *stiffness matrix* for the Laplace operator.

Above definitions of mass and stiffness matrices are special cases of *Riesz matrices* \mathbf{R}_Z for spaces Z , which fulfil the equality

$$\|v\|_Z^2 = \mathbf{c}^T \mathbf{R}_Z \mathbf{c}. \quad (12)$$

Some examples for functions $v \in S_j \subset Z$ are

$$\mathbf{R}_{L_2} = \mathbf{M}_j, \quad \mathbf{R}_{H^1} = \mathbf{M}_j + \mathbf{A}_j.$$

Extensions to Sobolev norms of higher integral order are straightforward, provided that the basis functions Φ_j are sufficiently smooth.

In the following, it will be at times useful to represent operators over a redundant set of functions. Using (7), we denote the resulting (positive semidefinite) matrices by

$$\bar{\mathbf{M}}_j := (\Theta_j, \Theta_j) = \mathbf{H}_j^T \mathbf{M}_j \mathbf{H}_j, \quad \bar{\mathbf{A}}_j := (\nabla \Theta_j, \nabla \Theta_j) = \mathbf{H}_j^T \mathbf{A}_j \mathbf{H}_j, \quad \text{etc.} \quad (13)$$

The goal of this paper is to construct Riesz matrices for Sobolev spaces of arbitrary real smoothness H^s , $s \in \mathbb{R}$, for a possibly large range of $s \in (-\tilde{\gamma}, \gamma)$. For finite element bases, we cover the case $s \geq 0$, while for biorthogonal wavelet bases, the whole range $s \in \mathbb{R}$ is in principle accessible.

However, it is an important point that fractional Sobolev norms, $s \notin \mathbb{Z}$, are only defined up to equivalence [1]. On the one hand, this implies that unique Riesz matrices do not exist for such s . On the other hand, this guarantees some freedom in the actual construction, since equivalence is all that can reasonably be expected.

These considerations motivate the following postulates for the construction of Riesz matrices \mathbf{R}_{H^s} over a range $s \in [0, \gamma)$ (for finite elements) or $s \in (-\tilde{\gamma}, \gamma)$ (for wavelets), which compute Sobolev norms numerically according to (12). These have before been formulated in [10].

Conclusion 2.2. We aim to construct Riesz operators with the following properties.

1. Norms of integral smoothness must be computed exactly.

2. Norms of fractional smoothness must be computed up to equivalence.
3. The norm of constant functions must be computed exactly for any s .
4. The computations up to any fixed level j must require only $\mathcal{O}(N_j)$ operations.

The first two criteria are important for applications in optimization and control: Exactness for the integral case ensures comparability between different implementations or problems, while the postulate of equivalence is fundamental for the treatment of fractional smoothnesses. The third criterion implements a certain degree of numerical consistency, while the fourth is clearly indispensable for algorithms of optimal computational complexity. We will in the following provide constructions which satisfy all of the above requirements.

2.3. Norm equivalences. The existence of norm equivalences between Sobolev spaces H^s and weighted multilevel decompositions provides – among many other uses – the basis for optimal preconditioners of elliptic PDEs and the construction of biorthogonal wavelets. They also play a fundamental role in the construction of Riesz operators discussed here. Therefore, we shortly comment on the main mathematical ideas, which can be expressed in the general framework of biorthogonal space decompositions, see e.g. [12, 17, 18, 22].

Definition 2.3. The collections of functions $\Phi_j \subset S_j$ and $\tilde{\Phi}_j \subset S_j$ are called biorthogonal if

$$\langle \Phi, \tilde{\Phi} \rangle = \mathbf{I}.$$

The collection $\tilde{\Phi}_j$ is also called the *dual basis* of Φ_j .

Theorem 2.4. Let $S_{j_0} \subset S_{j_0+1} \subset \dots$ and $\tilde{S}_{j_0} \subset \tilde{S}_{j_0+1} \subset \dots$ be two multiresolution sequences of $\mathcal{H} = L_2(\Omega)$. Let Φ_j and $\tilde{\Phi}_j$ be biorthogonal, uniformly stable bases of the nested spaces S_j and \tilde{S}_j , respectively.

(a) Then the projectors defined by

$$Q_j v := \langle v, \tilde{\Phi}_j \rangle \Phi_j, \quad \tilde{Q}_j \tilde{v} := \langle \tilde{v}, \Phi_j \rangle \tilde{\Phi}_j \quad (14)$$

are uniformly bounded in L_2 with $\text{Im}(Q_j) = S_j$ and $\text{Im}(I - Q_j) = (\tilde{S}_j)^{\perp_{L_2}}$. For the adjoint projectors $(\tilde{Q}_j)_j$ holds analogously $\text{Im}(\tilde{Q}_j) = \tilde{S}_j$ and $\text{Im}(I - \tilde{Q}_j) = (S_j)^{\perp_{L_2}}$.

(b) Under the extra assumption that there exist direct and inverse estimates up to smoothness $0 < \gamma < d$ for the primal subspaces S_j and up to smoothness $0 < \tilde{\gamma} < \tilde{d}$ for the \tilde{S}_j , the following norm equivalence emerges,

$$\|v\|_{H^s}^2 \sim \sum_{j=j_0}^{\infty} \rho^{2js} \| (Q_j - Q_{j-1}) v \|_{L_2}^2, \quad v \in H^s, s \in (-\tilde{\gamma}, \gamma). \quad (15)$$

Analogous results hold for interchanged roles of (γ, d) and $(\tilde{\gamma}, \tilde{d})$, with Q_j replaced by \tilde{Q}_j .

To ensure a consistent notation, we use the convention $X_{j_0-1} := 0$ for any operator, vector or matrix X unless stated otherwise. It is convenient to express the norm equivalence in terms of Sobolev shift operators Σ_s [18],

$$\Sigma_s v := \sum_{j=j_0}^{\infty} \rho^{(j-j_0)s} (Q_j - Q_{j-1}) v, \quad \|\Sigma_t v\|_{H^s} \sim \|v\|_{H^{s+t}}, \quad s, s+t \in (-\tilde{\gamma}, \gamma). \quad (16)$$

With these definitions, it follows that

$$\Sigma_0 = \text{id} \quad \text{and} \quad \Sigma_s^{-1} = \Sigma_{-s}. \quad (17)$$

For the multiresolution analysis introduced in Section 2.1, we identify $W_j = \text{Im}(Q_{j+1} - Q_j)$, cf. (8), and its dual counterpart $\tilde{W}_j = \text{Im}(\tilde{Q}_{j+1} - \tilde{Q}_j)$. Consequently,

$$W_j = (\tilde{S}_j)^{\perp_{L_2}} \cap S_{j+1} \quad \text{and} \quad \tilde{W}_j = (S_j)^{\perp_{L_2}} \cap \tilde{S}_{j+1}.$$

It follows trivially that the W_j and \tilde{W}_j are biorthogonal between levels,

$$W_{j_1} \perp \tilde{W}_{j_2} \quad \text{for} \quad j_1 \neq j_2. \quad (18)$$

Therefore, the collection of spaces W_j, \tilde{W}_j is called a biorthogonal decomposition of L_2 . This includes the spaces $W_{j_0-1} := S_{j_0}$ and $\tilde{W}_{j_0-1} := \tilde{S}_{j_0}$.

In the remainder of this document, we will employ this formalism for the construction of Riesz operators which satisfy the requirements motivated in the introduction. This will be done in two different settings, namely for standard finite elements and biorthogonal B-spline wavelets.

3. Evaluation with Finite Elements. The formalism of biorthogonal space decompositions covers standard finite elements as a special case. For example, the optimality of BPX-type preconditioners [9, 41, 42] can be derived using Theorem 2.4 for $s \in [0, \gamma]$ [19]. We will use a similar reasoning in this section to make use of the norm equivalences in the finite element setting. However, the crucial part here is *exactness* for integral order of smoothness, which is not important for preconditioning. In this sense, the following derivations are by necessity sharper than the cited results. Here we obtain a construction of Riesz operators which fulfills the requirements from Conclusion 2.2 for non-negative orders of smoothness.

3.1. Adapting the framework to finite elements. Let us assume that Φ_j is a piecewise polynomial, uniformly stable and local finite element basis of order d , which implies that $\gamma = d - \frac{1}{2}$. This fits into the biorthogonal framework by setting

$$\tilde{S}_j := S_j, \quad \tilde{\Phi}_j := \mathbf{M}_j^{-1} \Phi_j.$$

It follows that Φ_j and $\tilde{\Phi}_j$ are biorthogonal (however, this definition of $\tilde{\Phi}_j$ does not yield local functions), and that $\tilde{W}_j = W_j$. The projectors have the form

$$Q_j = \tilde{Q}_j = (\cdot, \Phi_j)_{L_2} \mathbf{M}_j^{-1} \Phi_j. \quad (19)$$

The uniform stability of Φ_j implies that also $\tilde{\Phi}_j$ is uniformly stable. Thus, all prerequisites for Theorem 2.4 are satisfied, and the norm equivalence (15) is valid for at least $s \in [0, \gamma]$.

A basic ingredient of the norm equivalence is the projection of a function $v \in S_J$ onto a complement space, $(Q_j - Q_{j-1})v$, with $j \leq J$. For any given expansion $v = \mathbf{c}^T \Phi_J$, the result can be computed by use of (5), (6), (11) and (19) as

$$\begin{aligned} (Q_j - Q_{j-1})v &= (\mathbf{c}^T \Phi_J, \Phi_j)_{L_2} \mathbf{M}_j^{-1} \Phi_j - (\mathbf{c}^T \Phi_J, \Phi_{j-1})_{L_2} \mathbf{M}_{j-1}^{-1} \Phi_{j-1} \\ &= \mathbf{c}^T \mathbf{M}_J (\mathbf{T}_{J,j} \mathbf{M}_j^{-1} - \mathbf{T}_{J,j-1} \mathbf{M}_{j-1}^{-1} \mathbf{G}_{j-1}^T) \Phi_j \\ &=: \mathbf{c}^T \mathbf{F}_{J,j} \Phi_j =: \mathbf{d}_{J,j}^T \Phi_j, \end{aligned} \quad (20)$$

where we have abbreviated the projection matrix onto the complement space as $\mathbf{F}_{J,j}$ and the projected coefficient vector as $\mathbf{d}_{J,j}$. With (15), this representation enables

us to calculate the H^s norm of v up to equivalence,

$$\|v\|_{H^s}^2 \sim \sum_{j=j_0}^J \rho^{2js} \| (Q_j - Q_{j-1}) v \|_{L_2}^2 = \sum_{j=j_0}^J \rho^{2js} \| \mathbf{d}_{J,j}^T \Phi_j \|_{L_2}^2 = \sum_{j=j_0}^J \rho^{2js} \mathbf{d}_{J,j}^T \mathbf{M}_j \mathbf{d}_{J,j}. \quad (21)$$

Because of the orthogonality of the complement spaces (18), it follows that

$$(\mathbf{d}_{J,j_1}^T \Phi_{j_1}, \mathbf{d}_{J,j_2}^T \Phi_{j_2})_{L_2} = 0 \quad \text{for } j_1 \neq j_2.$$

Defining the diagonal scaling matrix $\bar{\mathbf{D}}_j$ for the basis Θ_j by

$$\bar{\mathbf{D}}_j := \text{diag}(\rho^0 \mathbf{I}_{\Delta_{j_0}}, \rho^1 \mathbf{I}_{\Delta_{j_0+1}}, \dots, \rho^{J-j_0} \mathbf{I}_{\Delta_j})$$

and the cumulative scaled coefficient vector $\mathbf{e}_{J,s}$ by

$$\mathbf{e}_{J,s}^T := (\rho^0 \mathbf{d}_{J,j_0}^T, \rho^s \mathbf{d}_{J,j_0+1}^T, \dots, \rho^{(J-j_0)s} \mathbf{d}_{J,J}^T) = (\mathbf{d}_{J,j_0}^T, \dots, \mathbf{d}_{J,J}^T) \bar{\mathbf{D}}_J^s, \quad (22)$$

we have collected all ingredients for our first result.

Theorem 3.1. *The norm $\|v\|_{0,s}$ for $v = \mathbf{c}^T \Phi_J$, defined by*

$$\|v\|_{0,s}^2 := \mathbf{e}_{J,s}^T \bar{\mathbf{M}}_J \mathbf{e}_{J,s}, \quad (23)$$

is equal to the L_2 norm for $s = 0$ and for constant functions, and equivalent to the norm on H^s for $s \in [0, \gamma]$. It can be computed up to discretization error accuracy in $\mathcal{O}(N_J)$ operations.

Proof. The uniform scaling of $\mathbf{e}_{J,s}$ with $\rho^{-j_0 s}$ preserves the equivalence from (21). Exactness for $s = 0$ is ensured by (17). The norm for constant functions is always computed correctly since in this case only the first component $\rho^0 \mathbf{d}_{J,j_0}^T$ is nonzero, independent of s .

All matrices have a special, uniformly sparse structure and can therefore be applied in $\mathcal{O}(N_J)$ operations. Since $\mathbf{T}_{J,J} = \mathbf{I}$, the mass matrices on the highest level J cancel out. During the evaluation of $\mathbf{F}_{J,j}$, the inverses of the lower level mass matrices \mathbf{M}_j , $j < J$ must be applied. If these do not have a reasonably sparse direct factorization, they can be inverted numerically. Because of their sparsity and uniformly bounded condition number, an iterative solver such as the method of conjugate gradients, combined with a nested iteration strategy, delivers the result up to discretization error accuracy ρ^{-jd} in at most $\mathcal{O}(N_j)$ operations [11, 31]. A geometric series argument ensures this bound for the cumulative sum of all $\mathbf{F}_{J,j}$ over the range of levels j . \square

3.2. A unified scheme for norms of higher order. While Theorem 3.1 yields exactness for $s = 0$, it only guarantees equivalence for integer $s \in \mathbb{N}$. Therefore, we develop an analogous construction which is exact for $s = 1$. This can be repeated for all $s = 2, 3, \dots, s < \gamma$, resulting in a scale of exact integer Sobolev norms. For $s = 1$, let $v = \mathbf{c}^T \Phi_J$ be an arbitrary function in S_j . Using the operator Σ_s from (16), we obtain

$$\|v\|_{H^s}^2 \sim \|\Sigma_{s-1} v\|_{H^1}^2. \quad (24)$$

The term on the right hand side can be computed via

$$\Sigma_{s-1} v = \sum_{j=j_0}^J \rho^{(j-j_0)(s-1)} (Q_j - Q_{j-1}) v = \sum_{j=j_0}^J \rho^{(j-j_0)(s-1)} \mathbf{d}_{J,j}^T \Phi_j = \mathbf{e}_{J,s-1}^T \Theta_J, \quad (25)$$

where we have used the definitions from (20) and (22).

Theorem 3.2. *The norm $\|v\|_{1,s}$ for $v = \mathbf{c}^T \Phi_J$, defined by*

$$\|v\|_{1,s}^2 := \mathbf{e}_{J,s-1}^T (\bar{\mathbf{M}}_J + \bar{\mathbf{A}}_J) \mathbf{e}_{J,s-1}, \quad (26)$$

is equal to the H^1 norm for $s = 1$ and for constant functions, and equivalent to the norm on H^s for $s \in [0, \gamma]$. It can be computed up to discretization error accuracy in $\mathcal{O}(N_J)$ operations.

Proof. Equivalence to the H^s norm is assured by construction (24). The expression above is obtained using (25) and the definition of the extended mass and stiffness matrices (13). Exactness for general functions and $s = 1$ follows from (17). For constant functions, the contribution of $\bar{\mathbf{A}}$ vanishes and the situation is identical as in Theorem 3.1. Also, the reasoning for a computation time in $\mathcal{O}(N_J)$ is analogous. \square

Based on these results, we obtain a unified formulation of a norm for $[0, 1]$ which is exact for $s = 0$ and $s = 1$ and equivalent in between by a convex combination of $\|\cdot\|_{0,s}$ and $\|\cdot\|_{1,s}$. We generalize this ansatz for integer s as follows.

Proposition 3.3. *Suppose that there exist norms $\|\cdot\|_{i,s}$ for $i \in \{0, 1, \dots, S\}$, with $S < \gamma$, which fulfil $\|\cdot\|_{i,i} = \|\cdot\|_{H^i}$ and $\|\cdot\|_{i,s} \sim \|\cdot\|_{H^s}$ for $|s - i| \leq 1$, $0 \leq s < \gamma$. Moreover, they shall be exact for constant functions and computable with an effort of $\mathcal{O}(N_J)$. With the following definition of the hat functions $h_i(x)$ centered at i ,*

$$h_i(x) := \begin{cases} x - (i-1) & \text{for } i-1 \leq x < i, \\ (i+1) - x & \text{for } i \leq x \leq i+1, \\ 0 & \text{else,} \end{cases} \quad (27)$$

it follows that the norm

$$\|\cdot\|_s^2 := \sum_{i=0}^S h_i(s) \|\cdot\|_{i,s}^2 \quad (28)$$

is equal to Sobolev norms for integer smoothness $s = 0, 1, \dots, S$ and equivalent for $0 \leq s \leq S$. Furthermore, it is exact for constant functions and computable with an effort of $\mathcal{O}(N_J)$.

To illuminate some characteristics of this construction, we rewrite it in a slightly different form. To this end, define the scaled Riesz matrices on the redundant set,

$$\bar{\mathbf{R}}_0 := \bar{\mathbf{M}}_J, \quad \bar{\mathbf{R}}_1 := \bar{\mathbf{D}}_J^{-1} (\bar{\mathbf{M}}_J + \bar{\mathbf{A}}_J) \bar{\mathbf{D}}_J^{-1}, \quad \text{etc.} \quad (29)$$

using the notation from (12), (13). Then, using the definitions (23) and (26), equation (28) takes the form

$$\|v\|_s^2 = \mathbf{e}_{J,s}^T \bar{\mathbf{R}}_{J,s} \mathbf{e}_{J,s} := \mathbf{e}_{J,s}^T \left(\sum_{i=0}^S h_i(s) \bar{\mathbf{R}}_i \right) \mathbf{e}_{J,s} \quad (30)$$

with a newly defined unified Riesz operator $\bar{\mathbf{R}}_{J,s}$.

Conclusion 3.4. We close this section with the following considerations, which apply to the numerical scheme for the evaluation of Sobolev norms just presented above.

- As can be inferred from (30), the vectors $\mathbf{d}_{J,j}$ need only be computed once for each j and can be reused for the scaling operations (22). Therefore, single applications of the matrices \mathbf{M}_J and $(\mathbf{T}_{J,j} \mathbf{M}_j^{-1})^T$, $j_0 \leq j < J$, suffice.

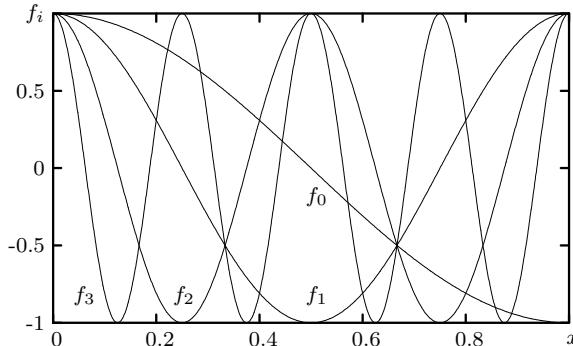
- The single scaled Riesz matrices from (29) are uniformly well-conditioned (when zero eigenvalues are ignored). This is a consequence of the integer norm equivalences, and can also be seen from a classical BPX preconditioning argument.
- An analogous statement for the convex combination $\bar{\mathbf{R}}_{J,s}$ (30) does not follow in an obvious way. The reason is that the eigensystems of the Riesz operators $\bar{\mathbf{R}}_i$ are different, and the number of zero eigenvalues is not preserved.
- In summary, we have established a family of Sobolev norms which fulfil the criteria from Conclusion 2.2. Explicit representations for $s = 0$ and $s = 1$ have been provided in (23) and (26), while the extension to higher orders of smoothness is possible with the techniques described above.
- While the computations described here can be straightforwardly implemented for uniform finite element discretizations, it is not obvious whether linear computational complexity can be maintained for adaptive finite elements. This is a major motivation to advance to wavelets as described in Section 4.

3.3. Numerical examples. To illustrate the evaluation scheme described above, we have conducted several computations for a number of sample functions and smoothness indices. The functions $f_i(x) := \cos(2^i \pi x)$, $i = 0, \dots, 3$ and their norms for several parameters are shown in Table 1. For each function, we have created a table where the rows correspond to increasing levels of resolution j , and the columns to the five smoothness indices $0, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}$ and 1 . The exact values for the L_2 and H^1 norms are provided in the bottom row. First of all, it can be seen that the computed L_2 and H^1 norms converge to the exact values in all cases. Convergence starts later for the more oscillatory functions, which makes sense since their periods are multiples of the grid size for low resolutions. The norms for non-integer smoothness interpolate between the extreme cases $s = 0$ and $s = 1$.

We notice also that the change of the norm between $s = 0$ and $s = \frac{1}{3}$ is the most extreme. This is allowed, since only equivalence is required for non-integer s . Yet, it indicates that there may be ways to improve the dependence of the construction on the smoothness parameter s .

A second set of results is shown in Table 2. The function g_1 has been chosen as an asymmetric hat function with peak at $\frac{1}{3}$, and we have set $g_2(x) := \sqrt{x}$. The results for g_1 and g_2 are similar to the results for the cosine functions f_i . For the square root function g_2 , where the H^1 norm is not finite, it can be seen that the corresponding values grow with the resolution. In summary, we may say that the numerical studies confirm the predicted behavior in all cases.

4. Evaluation with wavelets. In this section, we will cover the construction of Riesz operators for biorthogonal wavelets, where both the primal and the dual basis have compact support. Compared to standard finite elements, the process of biorthogonalization introduces additional complexity in the construction of the wavelet bases. However, once this is done, the resulting framework is perfectly symmetric between the primal and the dual side, which means that Sobolev norms of negative smoothness indices can be evaluated in much the same way as for positive smoothness. In particular, the mass matrices which are present in the finite element context drop out completely, which leads to a simpler structure of the projectors and the relations between different levels of resolution. Moreover, the framework can be generalized to adaptive wavelet discretizations using the ideas described in [15].



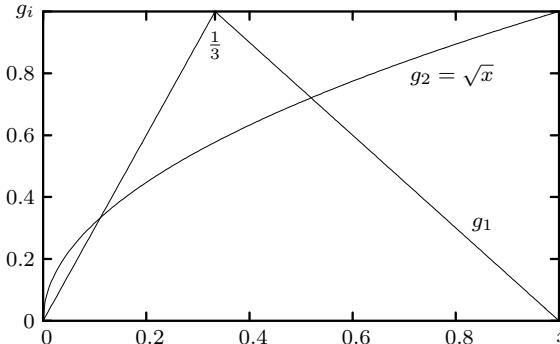
$j \setminus s$	0	1/3	1/2	3/4	1	$j \setminus s$	0	1/3	1/2	3/4	1
0	0.333	1.67	2.33	3.33	4.33	0	1	1	1	1	1
1	0.333	1.67	2.33	3.33	4.33	1	0.333	2.51	4.42	8.90	16.3
2	0.451	2.09	2.87	4.00	5.14	2	0.333	2.51	4.42	8.90	16.3
3	0.487	2.23	3.04	4.20	5.36	3	0.451	3.14	5.40	10.6	19.2
4	0.497	2.26	3.09	4.25	5.42	4	0.487	3.34	5.72	11.1	20.0
5	0.499	2.28	3.10	4.27	5.43	5	0.497	3.40	5.81	11.2	20.1
7	0.500	2.28	3.11	4.27	5.43	6	0.500	3.42	5.84	11.3	20.2
6	0.500	2.28	3.11	4.27	5.43	7	0.500	3.42	5.84	11.3	20.2
8	0.500	2.28	3.11	4.27	5.44	8	0.500	3.42	5.84	11.3	20.2
	1/2		$\ f_0\ _{H^s}^2$		5.44		1/2		$\ f_1\ _{H^s}^2$		20.2
$j \setminus s$	0	1/3	1/2	3/4	1	$j \setminus s$	0	1/3	1/2	3/4	1
0	1	1	1	1	1	0	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1
2	0.333	3.94	8.71	24.8	64.3	2	1	1	1	1	1
3	0.333	3.94	8.71	24.8	64.3	3	0.333	6.23	17.4	69.9	256.
4	0.451	4.91	10.6	29.5	75.4	4	0.333	6.23	17.4	69.9	256.
5	0.487	5.23	11.3	30.9	78.4	5	0.451	7.76	21.2	82.9	300.
6	0.497	5.32	11.4	31.3	79.2	6	0.487	8.27	22.4	86.9	312.
7	0.499	5.34	11.5	31.4	79.4	7	0.497	8.41	22.8	88.0	315.
8	0.500	5.35	11.5	31.4	79.4	8	0.499	8.45	22.9	88.3	316.
	1/2		$\ f_2\ _{H^s}^2$		79.5		1/2		$\ f_3\ _{H^s}^2$		316.

TABLE 1. We show several functions f_i and numerical results for norms of several smoothness indices s . These have been evaluated in the finite element setting.

4.1. Introduction to biorthogonal wavelets. Biorthogonal wavelets are based on primal and dual multiresolution analyses as described in Section 2. We assume the existence of bases Φ_j , Ψ_j and $\tilde{\Phi}_j$, $\tilde{\Psi}_j$ which are uniformly stable, local and of polynomial exactness of the order d and \tilde{d} , respectively. In addition, we demand biorthogonality, that is,

$$\langle \Phi_j, \tilde{\Phi}_j \rangle = \mathbf{I}, \quad \langle \Psi_j, \tilde{\Psi}_j \rangle = \mathbf{I}, \quad j = j_0, j_0 + 1, \dots \quad (31)$$

Existence of such bases on the real line has been shown in [16], where B-splines are used as primal functions and the dual functions are implicitly specified by recursion formula. These have later been modified to construct wavelets on the unit interval [21]. Wavelets where both primal and dual functions are piecewise polynomials have also been constructed [22, 37]. For practical purposes, several extensions and optimizations have been developed, see e.g. [3, 10, 36, 40].



$j \setminus s$	0	$1/3$	$1/2$	$3/4$	1	$j \setminus s$	0	$1/3$	$1/2$	$3/4$	1
0	0	0	0	0	0	0	0.333	0.667	0.833	1.08	1.33
1	0.188	0.494	0.762	1.39	2.44	1	0.451	0.808	0.995	1.29	1.62
2	0.281	0.662	1.02	1.92	3.66	2	0.486	0.813	1.00	1.35	1.83
3	0.319	0.761	1.17	2.21	4.26	3	0.496	0.796	0.982	1.36	2.01
4	0.330	0.789	1.21	2.31	4.55	4	0.499	0.780	0.961	1.36	2.19
5	0.332	0.797	1.23	2.35	4.70	5	0.500	0.770	0.945	1.36	2.36
6	0.333	0.799	1.23	2.37	4.76	6	0.500	0.764	0.935	1.35	2.54
7	0.333	0.800	1.23	2.37	4.80	7	0.500	0.761	0.929	1.35	2.71
8	0.333	0.800	1.23	2.37	4.82	8	0.500	0.759	0.925	1.34	2.88
	$1/3$		$\ g_1\ _{H^s}^2$		$29/6$		$1/2$		$\ g_2\ _{H^s}^2$		∞

TABLE 2. These tables show fractional Sobolev norms for the two functions g_1 and g_2 , again computed in the finite element setting.

It will be convenient to abbreviate $\Psi_{j_0-1} := \Phi_j$, $\nabla_{j_0-1} := \Delta_j$, and to apply analogous definitions for the dual basis. Then the formalism allows to specify primal and dual multi-scale bases Ψ , $\tilde{\Psi}$ of infinite cardinality,

$$\Psi := \bigcup_{j=j_0-1}^{\infty} \Psi_j, \quad \tilde{\Psi} := \bigcup_{j=j_0-1}^{\infty} \tilde{\Psi}_j, \quad \langle \Psi, \tilde{\Psi} \rangle = \mathbf{I}.$$

For any fixed maximum level J , the two-scale transformation (10) can be iterated, which leads to a matrix $\mathbf{W}_j : \ell_2(\Delta_J) \rightarrow \ell_2(\Delta_J)$ operating between Φ_J and the multi-scale basis $\Psi_{(J)}$ up to that level,

$$\Psi_{(J)}^T := (\Psi_{j_0-1}^T, \dots, \Psi_{J-1}^T) = \Phi_J^T \mathbf{W}_J. \quad (32)$$

This *multi-scale transformation* \mathbf{W}_J is composed of two-scale operators,

$$\mathbf{W}_J := \mathbf{W}_{J,J-1} \cdots \mathbf{W}_{J,j_0} \quad \text{with} \quad \mathbf{W}_{J,j} := \begin{pmatrix} \mathbf{G}_j & 0 \\ 0 & \mathbf{I} \end{pmatrix}, \quad (33)$$

and can be applied in $\mathcal{O}(N_J)$ operations due to a geometric series argument. The two-scale and multi-scale transformations \mathbf{G}_j , \mathbf{W}_j resp. $\tilde{\mathbf{G}}_j$, $\tilde{\mathbf{W}}_j$ exist for both collections, see (5), (9). Biorthogonality (31) implies that $\mathbf{G}_j^T = \tilde{\mathbf{G}}_j^{-1}$ and $\mathbf{W}_j^T = \tilde{\mathbf{W}}_j^{-1}$.

Combining the projectors defined in (14) with the biorthogonality of the subspaces (18), the norm equivalence (15) can be restated in terms of the complement

(wavelet) bases $\Psi_j, \tilde{\Psi}_j$,

$$\|v\|_{H^s}^2 \sim \sum_{j=j_0-1}^{\infty} \rho^{2js} \|(v, \tilde{\Psi}_j)_{L_2}\|^2, \quad v \in H^s, s \in (-\tilde{\gamma}, \gamma). \quad (34)$$

The dual relation, obtained by switching the roles of Ψ_j and $\tilde{\Psi}_j$, holds for $s \in (-\gamma, \tilde{\gamma})$. The formulation (34) suggests to absorb the scaling factors ρ^{js} into the wavelet basis. To this end, we define the diagonal scaling

$$\begin{aligned} \Psi^s &:= \mathbf{D}^{-s} \Psi := \{\rho^{-(j-(j_0-1))s} \psi_{j,k}\}_{j \geq j_0-1, k \in \nabla_j}, \\ \tilde{\Psi}^s &:= \mathbf{D}^s \tilde{\Psi}, \quad \langle \Psi^s, \tilde{\Psi}^s \rangle = \mathbf{I}. \end{aligned} \quad (35)$$

By these definitions, the *Riesz basis property* for Ψ^s is established, i.e.,

$$\|v\|_{H^s} \sim \|\mathbf{d}\|, \quad H^s \ni v = \mathbf{d}^T \Psi^s, \quad s \in (-\tilde{\gamma}, \gamma). \quad (36)$$

4.2. Primal and dual integral Sobolev norms. Let a function be expanded in an unscaled wavelet basis according to $L_2 \ni v = \mathbf{d}^T \Psi$. The L_2 norm can be evaluated by

$$\|v\|_{L_2}^2 = (\mathbf{d}^T \Psi, \mathbf{d}^T \Psi)_{L_2} = \mathbf{d}^T (\Psi, \Psi)_{L_2} \mathbf{d} =: \mathbf{d}^T \mathbf{M}_\Psi \mathbf{d}, \quad (37)$$

with the (infinite-dimensional) mass matrix \mathbf{M}_Ψ in wavelet coordinates. The norm in H^1 is computed according to

$$\begin{aligned} \|v\|_{H^1}^2 &= \|v\|_{L_2}^2 + |v|_{H^1}^2 = \mathbf{d}^T \mathbf{M}_\Psi \mathbf{d} + (\nabla \mathbf{d}^T \Psi, \nabla \mathbf{d}^T \Psi)_{L_2} \\ &= \mathbf{d}^T (\mathbf{M}_\Psi + \mathbf{A}_\Psi) \mathbf{d}, \end{aligned}$$

with the definition of the stiffness matrix in wavelet coordinates, $\mathbf{A}_\Psi := (\nabla \Psi, \nabla \Psi)_{L_2}$.

When we expand v in the natural wavelet basis for H^1 according to (35), $H^1 \ni v = \mathbf{d}^T \Psi^1 = \mathbf{d}^T \mathbf{D}^{-1} \Psi$, we can write the H^1 norm as

$$\|v\|_1^2 = \mathbf{d}^T \mathbf{D}^{-1} (\mathbf{M}_\Psi + \mathbf{A}_\Psi) \mathbf{D}^{-1} \mathbf{d} =: \mathbf{d}^T \mathbf{R}_1 \mathbf{d}, \quad (38)$$

where we have hidden the definition of \mathbf{R}_1 , the Riesz matrix in the natural wavelet basis. Note that due to (36), \mathbf{R}_1 is uniformly well-conditioned. Furthermore, \mathbf{M}_Ψ , \mathbf{A}_Ψ and \mathbf{R}_1 are symmetric and positive definite. Norms of higher integral order $\mathbb{N}_0 \ni i < \gamma$ are computed analogously via Riesz matrices \mathbf{R}_i with $\kappa(\mathbf{R}_i) \sim 1$.

For practical applications, \mathbf{M}_Ψ and \mathbf{A}_Ψ are not applied directly. Instead, the fast wavelet transform from (32), (33) can be used via the identities

$$\mathbf{M}_{\Psi(j)} = \mathbf{W}_j^T \mathbf{M}_j \mathbf{W}_j, \quad \mathbf{A}_{\Psi(j)} = \mathbf{W}_j^T \mathbf{A}_j \mathbf{W}_j.$$

In the framework of biorthogonal wavelets, the evaluation of dual norms is done with relative ease. As an example of the main technique, we first assure that L_2 can be identified with its dual. To this end, we apply the general definition of dual norms,

$$\|v\|_{\mathcal{H}'} := \sup_{0 \neq w \in \mathcal{H}} \frac{\langle v, w \rangle}{\|w\|_{\mathcal{H}}}, \quad (39)$$

to the expansion $L_2' \ni \tilde{v} = \tilde{\mathbf{d}}^T \tilde{\Psi} =: \mathbf{d}^T \Psi$. For any $L_2 \ni w = \mathbf{w}^T \Psi$, we may write

$$\|\tilde{v}\|_{L_2'} = \sup_{0 \neq w \in L_2} \frac{\langle \tilde{v}, w \rangle}{\|w\|_{L_2}} = \sup_{0 \neq \mathbf{w} \in \ell_2} \frac{\langle \tilde{\mathbf{d}}^T \tilde{\Psi}, \mathbf{w}^T \Psi \rangle}{(\mathbf{w}^T \mathbf{M}_\Psi \mathbf{w})^{\frac{1}{2}}} = \sup_{0 \neq \mathbf{w} \in \ell_2} \frac{\tilde{\mathbf{d}}^T \mathbf{w}}{(\mathbf{w}^T \mathbf{M}_\Psi \mathbf{w})^{\frac{1}{2}}}. \quad (40)$$

Here we have used (37) and the biorthogonality condition (31). After all, the expression is formulated in vectors and matrices on ℓ_2 and can be reduced further by the substitution $\mathbf{g} := \mathbf{M}_\Psi^{1/2} \mathbf{w}$ as follows,

$$\|\tilde{v}\|_{L_2'} = \sup_{0 \neq \mathbf{g} \in \ell_2} \frac{(\mathbf{M}_\Psi^{-1/2} \tilde{\mathbf{d}})^T \mathbf{g}}{(\mathbf{g}^T \mathbf{g})^{1/2}} = (\tilde{\mathbf{d}}^T \mathbf{M}_\Psi^{-1} \tilde{\mathbf{d}})^{1/2} = (\mathbf{d}^T \mathbf{M}_\Psi \mathbf{d})^{1/2} = \|\tilde{v}\|_{L_2}. \quad (41)$$

From this identity, we conclude that biorthogonal wavelet expansions on L_2 are consistent with the fact that L_2 can be identified with its dual, and the norms are identical. Even when the expansion $v = \mathbf{d}^T \Psi$ is not available, the dual norm can be efficiently computed, as \mathbf{M}_Ψ is uniformly well-conditioned, and its inverse can be applied in $\mathcal{O}(N_J)$ operations for any given level J .

After these preparations, we discuss the norm of $H^{-1} = (H^1)'$. We expand the function $H^{-1} \ni \tilde{v} = \tilde{\mathbf{d}}^T \tilde{\Psi}^1$, and the trial function $H^1 \ni w = \mathbf{w}^T \Psi^1$. The numerator of (39) is derived analogously to (40),

$$\langle \tilde{v}, w \rangle = \langle \tilde{\mathbf{d}}^T \tilde{\Psi}^1, \mathbf{w}^T \Psi^1 \rangle = \tilde{\mathbf{d}}^T \langle \tilde{\Psi}^1, \Psi^1 \rangle \mathbf{w} = \tilde{\mathbf{d}}^T \mathbf{w}. \quad (42)$$

We have used here the general biorthogonality relation (35). With a similar substitution as in the derivation of (41), we arrive at

$$\|v\|_{H^{-1}}^2 = \tilde{\mathbf{d}}^T \mathbf{R}_1^{-1} \tilde{\mathbf{d}}.$$

Remark 4.1. Due to the uniformly bounded condition number of \mathbf{R}_1 , the norm $\|v\|_{H^{-1}}$ can be evaluated efficiently in the wavelet setting. In contrast, a standard finite element ansatz would necessitate the multilevel projections detailed in Section 3, and an additional mass matrix would need to be applied in (42). The framework of biorthogonal wavelets is thus particularly suited for the evaluation of positive and negative integral Sobolev norms. Norms of higher order can be processed in the same fashion as exercised here for H^{-1} , L_2 and H^1 , provided that the ansatz functions are sufficiently regular.

4.3. A unified scheme to evaluate arbitrary real norms. In this section, we put together all ingredients to establish a construction of Riesz matrices which fulfills the criteria from Conclusion 2.2. In order to compare our construction to existing approaches, we first give a very short review of the state of the art of the evaluation of general Sobolev norms in the wavelet framework, and comment on the deficits. To this end, let us consider the expansion of a function v in the unscaled wavelet basis, $v = \mathbf{d}^T \Psi$. In [20], the norm used is

$$\|v\|_{H^s}^{(1)} := (\mathbf{d}^T \mathbf{D}^{2s} \mathbf{d})^{1/2}. \quad (43)$$

This norm is always equivalent to $\|\cdot\|_{H^s}$, but never exact for $s \in \mathbb{Z}$. It corresponds to the simplest approximation $\mathbf{R}_s := \mathbf{I}$. The choice in [11] is

$$\|v\|_{H^s}^{(2)} := (\mathbf{d}^T \mathbf{D}^s \mathbf{M}_\Psi \mathbf{D}^s \mathbf{d})^{1/2},$$

which is always equivalent, but only exact for $s = 0$ (the L_2 case). This conforms with the approximation $\mathbf{R}_s := \mathbf{R}_0$ and can be seen as a qualitative improvement compared to (43). Anyway, both variants permit the evaluation of the corresponding dual norms up to equivalence, and they satisfy the fundamental requirement of computational efficiency.

To obtain a construction which is exact for *both* L_2 and H^1 , one might consider a multiplicative interpolation as follows,

$$\|v\|_{H^s}^{(3)} := (\mathbf{d}^T \mathbf{M}_\Psi^{\frac{1-s}{2}} (\mathbf{M}_\Psi + \mathbf{A}_\Psi)^s \mathbf{M}_\Psi^{\frac{1-s}{2}} \mathbf{d})^{1/2}.$$

This expression is motivated by the analogous definition for spectral elements, where it yields the norm in the sense of a Fourier decomposition for all $s \in \mathbb{R}$. In the context of wavelets, it is exact for $s = 0$ and $s = 1$, and equivalent in between. However, the fractional powers of matrices require a singular value decomposition, which is prohibitively expensive in practice. Moreover, constant functions are not preserved. Thus, it seems that this ansatz is again not appropriate.

In summary, none of the above constructions based on wavelet discretizations meet the criteria from Conclusion 2.2. However, it turns out that the interpolation by convex combination which we devised in Section 3 can be generalized to the wavelet setting.

To this end, let \mathbf{d}_s be the coefficient vector of a function $v = \mathbf{d}_s^T \Psi^s$ in a natural wavelet basis Ψ^s for H^s , and suppose that there exist representations \mathbf{R}_i for $i \in \{0, 1, \dots, S\}$, with $S < \gamma$, which fulfil

$$\|v\|_{H^i}^2 = \mathbf{d}_i^T \mathbf{R}_i \mathbf{d}_i, \quad \|v\|_{H^s}^2 \sim \mathbf{d}_s^T \mathbf{R}_i \mathbf{d}_s \quad \text{for } |s - i| \leq 1, 0 \leq s < \gamma. \quad (44)$$

Examples are $\mathbf{R}_0 = \mathbf{M}_\Psi$ and \mathbf{R}_1 as defined in (38). Higher orders can be derived from the standard definition of Sobolev norms. We know that all \mathbf{R}_i are symmetric positive definite and spectrally equivalent to the identity matrix. With the definition of the hat function $h_i(x)$ from (27), we formulate our proposal for a unified Riesz matrix within the wavelet framework.

Theorem 4.2. *For $s \in \mathbb{R}$ with $0 \leq s \leq S$, the norm defined by*

$$\begin{aligned} H^s \ni v &= \mathbf{d}_s^T \Psi^s, \\ \|v\|_s^2 := \sum_{i=0}^S h_i(s) \mathbf{d}_s^T \mathbf{R}_i \mathbf{d}_s &= \mathbf{d}_s^T \left(\sum_{i=0}^S h_i(s) \mathbf{R}_i \right) \mathbf{d}_s =: \mathbf{d}_s^T \mathbf{R}_s \mathbf{d}_s, \end{aligned} \quad (45)$$

is equal to the standard Sobolev norms for integral s and equivalent for fractional s . It can be computed in linear time.

Proof. For $s \in \mathbb{N}_0$, only one summand survives with $i = s$, and the above formula reduces to the expression given in (44). For fractional s , the formulation on the right of (45) is a convex combination of two summands which are both spectrally equivalent to the identity matrix. Consequently, the whole sum is spectrally equivalent to the identity matrix, and the expression yields a norm which is equivalent to the corresponding fractional Sobolev norm. Likewise, the computational efficiency and the exactness for constant functions is inherited as a feature of the matrices \mathbf{R}_i . \square

Thus, the general Riesz matrix \mathbf{R}_s has a uniformly bounded condition number. Therefore, its inverse can be computed numerically with an effort of $\mathcal{O}(N_J)$, where J is any given maximum level of resolution, by conjugate gradient iterations combined with a nested iteration ansatz [10]. This allows to evaluate dual norms by a generalization of the considerations in Section 4.2.

Corollary 4.3. *For an expansion in the natural dual basis $\tilde{\Psi}^s$ of H^{-s} , $0 \leq s < \min\{\tilde{\gamma}, \gamma\}$, the dual norm can be computed by inverting \mathbf{R}_s , i.e.,*

$$H^{-s} \ni \tilde{v} = \tilde{\mathbf{d}}_s^T \tilde{\Psi}^s, \quad \|\tilde{v}\|_{-s}^2 = \tilde{\mathbf{d}}_s^T \mathbf{R}_s^{-1} \tilde{\mathbf{d}}_s.$$

This expression is exact for integer s , and equivalent for $s \in \mathbb{R}$. Its numerical computation up to discretization error accuracy requires an effort linear in the amount of degrees of freedom.

$j \setminus s$	0	1/3	1/2	3/4	1	$j \setminus s$	0	1/3	1/2	3/4	1
3	0.487	2.11	2.92	4.14	5.36	3	0.500	0.115	0.0833	0.0588	0.0455
4	0.497	2.14	2.96	4.19	5.42	4	0.500	0.116	0.0840	0.0593	0.0459
5	0.499	2.15	2.97	4.20	5.43	5	0.500	0.116	0.0842	0.0595	0.0460
6	0.500	2.15	2.97	4.20	5.43	6	0.500	0.117	0.0842	0.0595	0.0460
7	0.500	2.15	2.97	4.20	5.43	7	0.500	0.117	0.0842	0.0595	0.0460
8	0.500	2.15	2.97	4.20	5.43	8	0.500	0.117	0.0842	0.0595	0.0460
	1/2		$\ f_0\ _{H^s}^2$		5.44		1/2		$\ f_0\ _{(H^s)'}^2$		0.0460

TABLE 3. These listings show primal and dual Sobolev norms of f_0 computed in the wavelet setting, for different levels of resolution j and smoothness indices s .

$j \setminus s$	0	1/3	1/2	3/4	1	$j \setminus s$	0	1/3	1/2	3/4	1
3	0.333	85.7	128.	192.	256.	3	0.333	85.7	128.	192.	256.
4	0.333	85.7	128.	192.	256.	4	0.333	85.7	128.	192.	256.
5	0.451	103.	153.	226.	300.	5	0.451	103.	153.	226.	300.
6	0.487	110.	162.	236.	312.	6	0.487	110.	162.	236.	312.
7	0.497	112.	164.	239.	315.	7	0.497	112.	164.	239.	315.
8	0.499	112.	165.	240.	316.	8	0.499	112.	165.	240.	316.
	1/2		$\ f_3\ _{H^s}^2$		316.		1/2		$\ f_3\ _{(H^s)'}^2$		0.000790
$j \setminus s$	0	1/3	1/2	3/4	1	$j \setminus s$	0	1/3	1/2	3/4	1
3	0.333	0.00130	0.00087	0.00058	0.000433	3	0.333	0.00130	0.00087	0.00058	0.000433
4	0.486	0.00190	0.00127	0.00084	0.000632	4	0.486	0.00190	0.00127	0.00084	0.000632
5	0.499	0.00243	0.00157	0.00102	0.000750	5	0.499	0.00243	0.00157	0.00102	0.000750
6	0.500	0.00253	0.00164	0.00106	0.000780	6	0.500	0.00253	0.00164	0.00106	0.000780
7	0.500	0.00256	0.00166	0.00107	0.000788	7	0.500	0.00256	0.00166	0.00107	0.000788
8	0.500	0.00257	0.00166	0.00107	0.000790	8	0.500	0.00257	0.00166	0.00107	0.000790
	1/2		$\ f_3\ _{(H^s)'}^2$		0.000790		1/2		$\ f_3\ _{(H^s)'}^2$		0.000790

TABLE 4. These tables show primal and dual Sobolev norms for the function f_3 , computed in the wavelet setting.

Indeed, compared to the finite element setting, wavelets permit the evaluation of primal and dual norms within an elegant formalism. This construction satisfies all criteria from Conclusion 2.2.

4.4. Numerical examples. In this section we collect results for the evaluation of primal and dual results using a biorthogonal spline wavelet basis. The wavelets are constructed as in [21] with some corrections as outlined in [10]. The lowest level for this basis is $j_0 = 3$. We have reused functions from the experiments with finite elements shown in Table 1 and 2. Within the wavelet framework, the function f_0 is examined in Table 3, f_3 in Table 4, and g_1 and g_2 in Table 5. For each row, the left listing contains the values for the primal norms, and the right listing those for the dual norms.

As required by the theory, for all functions the values for integer $s = 0$ and $s = 1$ in the primal case are equal to those for the evaluation with finite elements. They converge to the exact values with increasing resolution j . The primal norms increase and the dual norms decrease monotonously with s , as it should be.

There are two main differences between the cosine functions f_i and the asymmetric hat and square root functions g_1 and g_2 . Firstly, for the cosine functions there are significant changes in absolute value between the situations $s = 0$ and $s > 0$.

$j \setminus s$	0	1/3	1/2	3/4	1	$j \setminus s$	0	1/3	1/2	3/4	1
3	0.319	1.63	2.29	3.27	4.26	3	0.333	0.257	0.255	0.253	0.253
4	0.330	1.71	2.41	3.46	4.54	4	0.333	0.257	0.255	0.253	0.253
5	0.332	1.74	2.45	3.54	4.69	5	0.333	0.257	0.255	0.253	0.253
6	0.333	1.75	2.47	3.57	4.76	6	0.333	0.257	0.255	0.253	0.253
7	0.333	1.75	2.47	3.58	4.78	7	0.333	0.257	0.255	0.253	0.253
8	0.333	1.75	2.47	3.58	4.82	8	0.333	0.257	0.255	0.253	0.253
	1/3	$\ g_1\ _{H^s}^2$			29/6		1/3	$\ g_1\ _{(H^s)'}^2$			
$j \setminus s$	0	1/3	1/2	3/4	1	$j \setminus s$	0	1/3	1/2	3/4	1
3	0.496	1.002	1.25	1.63	2.01	3	0.500	0.455	0.452	0.449	0.448
4	0.499	1.011	1.28	1.71	2.19	4	0.500	0.456	0.453	0.450	0.449
5	0.500	0.990	1.27	1.74	2.36	5	0.500	0.457	0.453	0.450	0.449
6	0.500	0.966	1.24	1.75	2.54	6	0.500	0.457	0.453	0.450	0.449
7	0.500	0.949	1.22	1.75	2.71	7	0.500	0.457	0.453	0.450	0.449
8	0.500	0.938	1.20	1.75	2.88	8	0.500	0.457	0.453	0.451	0.449
	1/2	$\ g_2\ _{H^s}^2$			∞		1/2	$\ g_2\ _{(H^s)'}^2$			

TABLE 5. Here, numerical results for primal and dual Sobolev norms of the functions g_1 (top) and g_2 (bottom) are shown. Again, these results have been obtained using a wavelet discretization, with j the level of resolution.

These changes are larger than for the finite element version. Thus, the constants for the wavelet ansatz seem to be bigger compared to finite elements. Secondly, the dual norms of g_1 and g_2 do not change much with s . This may be related to their non-oscillatory nature which leads to only very small wavelet coefficients on the higher levels.

5. Conclusions. In this paper we propose a unified construction of Riesz matrices for both finite element and biorthogonal wavelet discretizations. The aim has been set to evaluate Sobolev norms computationally from given coefficient expansions. This task occurs in many recent applications in numerical simulation and optimization. However, the evaluation of fractional and dual Sobolev norms has to our knowledge not yet been systematically studied.

We provide an approach which computes integer norms exactly and interpolates equivalently for real smoothness indices. Norms of constant functions are always calculated exactly. The computational work is linear in the amount of unknowns. In addition, we specify a way to evaluate norms of negative order by a numerical inversion of Riesz operators in the framework of biorthogonal wavelets. Since these are uniformly well-conditioned, also dual norms can be computed in optimal complexity in the wavelet setting. Still, our approach contains some arbitrary elements which may be enhanced further or even be replaced by a conceptually different technique to tune the constants in the norm equivalences.

We reason that this approach is independent of the spatial dimensions. The fundamental requirement is the existence of direct and inverse estimates, which is met by a multitude of different constructions of finite elements and wavelets. While our construction is optimal up to a constant in terms of arithmetic operations for uniform finite element and wavelet discretizations, presently only the framework of biorthogonal wavelets seems to permit the extension to adaptive discretizations in optimal complexity.

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